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| Introduction | | | | | | | | | | | | | | | | | | | | | |
| Suppose X is a cts r.v. with pdf f(x) = | | | | | | | | | | | | | | | | | | | | = 1. c = 1/3. P(X > 0) = = 8/9 | |
| If X is a cts r.v. with pdf f(x) =  Find median of X. Find pdf of 5X + 10 | | | | | | F(y) = P(X ≤ y) = = -e-2x = 1 - e-2y, y ≥ 0. F(m) = 0.5. 1 - e-2m = 0.5. m = (ln 2)/2  Let W = 5X + 10. P(W ≤ w) = P(5X + 10 ≤ w) = P(X ≤ (w-10)/5), w ≥ 10 = 1 - e-2(w-10)/5  Pdf of W. fw(w) = P(W ≤ w) = e-2w/5 + 4, w ≥ 10 | | | | | | | | | | | | | | | |
| X is a cts r.v. with f(x) = | | | | | | | | | | | | | | | | | | | | = 1. c = 1. P(1/3 < X < 1/2) = = 1/6 | |
| If X ~ cdf F(x), pdf of F(X)? | | | | | | | | | | | | | | | | | | | | Let Y = F(X). FY(y) = P(Y ≤ y) = P(F(X) ≤ y) = P(X ≤ F-1(y)) = F(F-1(y)) = y | |
| Generate r.v. from pdf f(x) = | | | | | | | | | | | | | | | | | | | | F(x) = 1 - e-2x. F(X) = 1 - e-2X = U ~ uniform(0, 1). X = -ln(1-U)/2  1. Generate U ~ uniform(0,1). 2. Deliver X = -ln(1-U)/2 | |
| Expectation and Variance | | | | | | | | | | | | | | | | | | | | | |
| Find E(X) if X ~ pdf f(x) = | | | | | | | | | | | | | | | | | | | | E(X) = = . E(X2) = = . Var(X) = E(X2) - [E(X)]2 = | |
| If pdf of X, f(x) = , find E(X2) and E(-X) | | | | | | | | | | | | | E(X2) = = 1/3. E(-X) = = -1/2 | | | | | | | | |
| Lemma. If Y ≥ 0, E(Y) = | | | | Proof. = = = = = E(Y) | | | | | | | | | | | | | | | | | |
| If X ~ pdf f(x), then for any real-valued fn g, E[g(X)] = | | | | | | | Proof (for g(x) ≥ 0). E[g(X)] = =  = = = = | | | | | | | | | | | | | | |
| E(aX+b) = aE(X) + b | | | | | | | Proof. E(aX+B) = = a + b = aE(X) + b | | | | | | | | | | | | | | |
| Uniform r.v. | | | | | | | | | | | | | | | | | | | | | |
| If X ~ Uniform(0,1) what is pdf of (b-a)X + a? where a, b are constants and b > a  Find E((b-a)X + a) | | | | Let Y = (b-a)X + a. FY(y) = P(Y ≤ y) = P((b-a)X + a ≤ y) = P(X ≤ ) = = , a < y < b.  fY(y) = FY(y) = . E((b-a)X + a) = (b-a)E(X) + a = (b-a)(1/2) + a = (a+b)/2 OR E(Y) = | | | | | | | | | | | | | | | | | |
| X ~ Uniform(-1,1) = f(x) = | | | | | | | | | | | | | | | | | | | | P(|X| < 1/3) = P(-1/3 < x < 1/3) = = 1/3  P(X2 < 1/3) = P(-1/ < x < 1/) = = 1/ | |
| Normal r.v. | | | | | | | | | | | | | | | | | | | | | |
| Blood cholesterol level is approximately normally distributed with mean 220mg/dL and s.d. 15mg/dL. P(blood cholesterol level < 200)? | | | | | | | | | | | | | | | | | | | | Let X = blood cholesterol level. X ~ N(220, 152)  P(X < 200) = P( < ) = P(Z < -1.33) = 0.0918 | |
| If Y ~ Binomial(n = 1000, p = 0.3679), find P(Y ≥ 400)  = 1000\*0.3679 = 367.9. = 1000(0.3679)(1-0.3679) = 232.5496. = 15.2496 | | | | | | | | | | Exact: P(Y ≥ 400) = = 0.01954  Normal approximation: Y ~ N(367.9, 15.24962). P(Y ≥ 400) = P( ≥ ) = P(Z ≥ 2.07) = 1 - 0.9808 = 0.0192 | | | | | | | | | | | |
| If X ~ N (, ), then ~ N(0,1) | Proof. Let Y = . FY(y) = P(Y ≤ y) = P( ≤ y) = P(X ≤ + y) = FX( + y)  fY(y) = FY(y) = FX( + y) = fX( + y)( + y) = fX( + y) \* = \* = , -∞ < y < ∞. Y ~ N(0, 1) | | | | | | | | | | | | | | | | | | | | |
| Exponential r.v. | | | | | | | | | | | | | | | | | | | | | |
| Lifetime of a light bulb is exponential r.v. with mean 3 years. Let X = lifetime of light bulb. E(X) = 3. = . f(x) = e-x/3, x > 0. | | | | | | | | | | | | | | | | | | | | P(X < 3) = = 0.63 | |
| Show exponential r.v. is memoryless. X ~ Exp(). f(x) = , x > 0 | | | | | | | | | | | | | | | | | | P(X > s) = = . P(X > s+t) = = = P(X > s)P(X > t) | | | |
| Post office is staffed by 2 clerks. Suppose that when C enters office, he sees A being served by 1 clerk, B by the other. C service will begin when either A or B leaves. If amt of time clerk spend with customer is exponentially dist with parameter , what is the prob that C is last to leave? | | | | | | | | | | | | | | | | | | | | | By memoryless property, time clerk spend with C = time clerk spend with A or B (depending on who haven't finish). So prob = 1/2. |
| If X ~ Exp(), then E(X) = and Var(X) = | | | | | Proof. E(Xn) = = xn(-) + n (by parts) = = E(Xn-1)  E(X) = = . E(X2) = = E(X) = . Var(X) = - = | | | | | | | | | | | | | | | | |
| Other cts dist | | | | | | | | | | | | | | | | | | | | | |
| Γ(α) = (α - 1)Γ(α - 1)!, α > 1 | | Proof. Γ(α) = = -e-y + (by parts) = (-1) = (-1)Γ(α - 1) | | | | | | | | | | | | | | | | | | | |
| If α is an int, say α = n, then Γ(n) = (n-1)!. Note Γ(1) = = 1 | | | | | | | | | | | | | | | | | Γ(n) = (n-1)Γ(n-1) = ... = (n-1)(n-2)...(3)(2)(1)Γ(1) = (n-1)! | | | | |
| Xi ~ Exp(), i = 1,2,... and X­i are indep. Then Tn = X1 + X2 + ... + Xn ~ Gamma(n, )  Let Tn = time which nth event occurs, N(t) = num of events in time period [0,t]. Note {Tn ≤ t} = {N(t) ≥ n} | | | | | | | | | P(Tn ≤ t) = P(N(t) ≥ n) = . N(t) ~ Poisson(t)  P(Tn ≤ t) = . f(t) = P(Tn ≤ t) = = – = – = | | | | | | | | | | | | |
| X ~ Cauchy(θ). E(Xn) DNE for n = 1,2,... | | | | Consider θ = 0. E(X) = = = = ln(1+x2) = ∞ - ∞ = undefined | | | | | | | | | | | | | | | | | |
| Extra | | | | | | | | | | | | | | | | | | | | | |
| X is a cts r.v. w pdf f(x) = + for -1 < x < 3. | | | | | | | | | P(X > 0) = = . P(X2 < 1) = P(-1 < X < 1) = = . E(X) = = | | | | | | | | | | | | |
| A point is chosen at random on a line of length L. Interpret this statement and find prob ratio of shorter to longer segment is < 1/4 | | | | | | | Let X denote point chosen. X ~ Uniform(0, L). P(min(, ) < ) = 1 - P(min(, ) > ) = 1 - P( > , > ) = 1 - P(X > , X < ) = 1 - P( < X < ) = 1 - 3/5 = 2/5 | | | | | | | | | | | | | | |
| Arrived at bus stop at 10am. Bus arrive at some time uniformly dist btw 10am and 10.30 am. Let X = arrival of bus, X ~ U(0,30) | | | | | | | | | | | | | | | P(X ≥ 10) = = 2/3. P(X ≥ 25|X ≥ 15) = = = = 1/3 | | | | | | |
| Fire station is located along road of length a. If fire occur at points uniformly chosen on (0,a), where should station be to minimise expected dist from fire? i.e. choose t s.t. E(|X-t|) is minimized. | | | | | | | | | | | | | Let s(t) = E|X-t| = = + = . s'(t) = .  Let s'(t) = 0, then t = a/2. Checking, s''(a/2) > 0, so t = a/2 gives min value | | | | | | | | |
| Let X be a r.v..Let a be a real num, show E(X-a)2 = var(X) + [a-E(X)]2. What is min value of E(X-a)2? | | | | | | | | E(X-a)2 = E((X-) + ( - a))2 = E((X-)2 + 2(X -)( - a) + ( - a)2) = E(X-)2 + 2( - a)E(X -) + ( - a)2 = var(X) + (a-E(X))2. Min value of E(X-a)2 = var(X) and occurs at a = E(X) | | | | | | | | | | | | | |
| Let Z be standard normal r.v.. For x > 0, show P(Z > x) = P(Z < -x), P(|Z| > x) = 2P(Z > x), P(|Z| < x) = 2P(Z < x) -1 | | | | | | Note -Z also standard normal r.v.. P(Z > x) = P(-Z < -x) = P(Z < -x). P(|Z| > x) = P(Z > x) + P(Z < -x) = 2P(Z > x).  P(|Z| < x) = P(-x < Z < x) = P(Z < x) - P(Z ≤ -x) = P(Z < x) - P(Z ≥ x) = P(Z < x) - [1 - P(Z < x)] = 2P(Z < x) - 1 | | | | | | | | | | | | | | | |
| If X is normal r.v. w = 10 and = 36, compute P(X > 5), P(4 < X < 16) | | | | Y = (X-10)/6 ~ N(0,1). P(X ≥ 5) = P(Y ≥ -5/6) = P(Y < 5/6) = 0.7967. P ($ < X < 16) = P(-1 < Y < 1) = 2P(Y < 1) - 1 = 0.6826. | | | | | | | | | | | | | | | | | |
| Annual rainfall is normally distributed with = 40 and = 16. Prob that it will take over 10 years before a year occurs with rainfall > 50 inches. Assumptions made? | | | | | | | | | | | | In a year, P(X > 50) = P( > ) = P(Z > 2.5) = 1-0.99379  Assume events of observing rainfall greater than 50 inches in each year is indep. Then waiting time T until year w rainfall > 50 inches is geometric r.v. w p = 1 - 0.99379. P(T > 10) = (1-p)10 | | | | | | | | | |
| 1000 indep rolls of a fair die is made. Compute approximation to prob that num 6 will appear btw 150 and 200 times inclusively. If num 6 appears exactly 200 times, find prob that num 5 appear less than 150 times | | | | | | | | | | | Total num of sixes rolled, X ~ Binomial(1000,1/6). Using normal approximation, X ~ N(np, npq) = (1000/6, 5000/36). P(150 ≤ X ≤ 200) = P(149.5 < X < 200.5) ≈ P(-1.46 < Z < 2.87) = 0.9258  If 6 appear 200 times, prob 5 appear on other 800 rolls is 1/5. Y ~ Binomial(800, 1/5) ≈ N(800/5, 3200/25). P(Y < 150) = P(Y < 149.5) ≈ P(Z < (149.5-160)/sqrt(128)) = P(Z < - 0.93) = 0.1762 | | | | | | | | | | |
| In 10,000 indep toss, coin lands heads 5800 times. Is coin fair? | | | | Suppose coin is fair, then X ~ Binomial(10000,1/2). Using normal approximation, P(X ≥ 5800) = P(X ≥ 5799.5) ≈ P(Z ≥ (5799.5-5000)/sqrt(2500)) = P(Z ≥ 15.99) ≈ 0. So unlikely that coin is fair. | | | | | | | | | | | | | | | | | |
| Time required to repair machine is an exponentially dist r.v. w = 1/2. Prob that repair > 2 hours. Conditional prob repair takes 10 hours, given its duration exceeds 9 hours. | | | | | | | | | | | | | Let T denote repair time. P(T > 2) = = e-1  P(T > 10|T > 9) = P(T > 1) (by memoryless property) = e-1/2 | | | | | | | | |
| Y is exponentially dist r.v. w = 1. Compute pdf of r.v. X = logY | | | | | | | | | | | | | | | FX(x) = P(X ≤ x) = P(logY ≤ x) = P(Y ≤ ex) = FY(ex). fX(x) = fy(ex)ex = ex = | | | | | | |
| Weibull dist. Let > 0 and v . Suppose X is exponentially dist w mean 1. Find pdf and dist fn of Y where Y = + v | | | | Note that Y takes in value from (v, ∞). Let y > v, then FY(y) = P(Y ≤ y) = P( + v ≤ y) = P(X ≤ ) = FX() = 1 - . So, fY(y) = fx(). fY(y) = | | | | | | | | | | | | | | | | | |
| Let Y = . Show if X is a Weibull r.v. w params v, , then Y is an exponential r.v. w = 1. | | | | | | | | | P(Y ≤ y) = P(≤ y) = P(X ≤ v + ) = 1 - = 1 - e-y (cdf of exp w = 1) | | | | | | | | | | | | |
| Trains headed for A arrive at 15-mins intervals starting from 7am, while trains heading for B arrive at 15-mins intervals starting from 7.05am.  If passenger arrive at station at time uniformly dist btw 7 and 8am, and get on 1st train that arrives, prob go to A? If arrive from 7.10 to 8.10am? | | | | | | | | | | | | | | | | | | | P(A) = P(5 < X < 15 or 20 < X < 30 or 35 < X < 45 or 50 < X < 60) = 40/60 = 2/3 since X ~ Uniform(0,60)  P(A for 7.10 to 8.10) = P(10 < X < 15 or 20 < X < 30 or 35 < X < 45 or 50 < X < 60 or 65 < X < 70) = 2/3 | | |
| P(success) = .95. Approximate prob at most 10 of next 150 items produces are unacceptable. Let X denote num of unacceptable items among next 150 produced. | | | | | | | | | | | X ~ Binomial(150, 0.05) ≈ N(150\*0.5 = 7.5, 150\*.5\*.95 = 7.125)  P(X ≤ 10) = P(X ≤ 10.5) (continuity correction) = P(Z ≤ ) ≈ P(Z ≤ 1.1239) = .8695 | | | | | | | | | | |
| Curr price of stock is s. After 1 period, price is either us w prob p or ds w prob 1-p. Assume successive movements are indep, approximate prob stock price will be up at least 30% after next 1000 periods if u = 1.012, d = 0.990, p = .52. | | | | Let X = num of 1000 time periods in which stock increase. Price at end: suXd1000-X = sd1000(u/d)X  We need sd1000(u/d)X > 1.3s OR d1000(u/d)X > 1.3 OR X > = 469.2. Thus, we need ≥ 470 periods.  Using normal approximation for X ~ Binomial(1000, .52)  P(X ≥ 470) = P(X > 469.5) = P(Z > ) ≈ P(Z > -3.196) ≈ .9993 | | | | | | | | | | | | | | | | | |
| Show E[Y] = – | | | = = = = –  Similarly, = .  So, – = + = = E[Y] | | | | | | | | | | | | | | | | | | |
| Use the result that for a nonnegative r.v. Y, E[Y] = to show for a nonnegative r.v. X, E[Xn] = . | | | | | | | | | | | | | | | | Let t = xn, then = nxn-1. E[Xn] = = = | | | | | |
| Let X be a r.v. that takes on values btw 0 and c. i.e. P(0 ≤ X ≤ c) = 1. Show var(X) ≤ c2/4 | | | | Since 0 ≤ X ≤ c, then X2 ≤ cX, so E[X2] ≤ E[cX]  Var(X) = E[X2] - (E[X])2 ≤ E[cX] - (E[X])2 = cE[X] - (E[X])2 = E[X](c - E[X]) = c2[a(1-a)] (where a = E[X]/c) ≤ c2/4 (since max of a(1-a) = 1/4 using differentiation) | | | | | | | | | | | | | | | | | |
| If X is an exponential r.v. w mean 1/, show E[Xk] = , k = 1,2,... | | | | E[Xk] = = = = (1) (gamma pdf) = | | | | | | | | | | | | | | | | | |
| Show = . Let y = . = | | | | = = = = = P(Z > 0) = (1/2) = | | | | | | | | | | | | | | | | | |
| Find pdf of Y = eX when X is normally dist w params and . r.v. Y is said to have a lognormal dist w params and . | | | | | | | | | | | | | | FY(x) = P(Y ≤ x) = P(eX ≤ x) = P(X ≤ ln x) = FX(ln x).  fY(x) = fX(ln x)(1/x) = | | | | | | | |

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| Joint Distribution Fn | | | | | | | | | | | | |
| Marginal cdf of X, FX(x) = | | | Proof. FX(x) = P(X ≤ x) = P(X ≤ x, Y < ∞) = P() = = | | | | | | | | | |
| P(X > a, Y > b) = 1 - FX(a) - Fy(b) + F(a,b) | | | Proof. P(X > a, Y > b) = 1 - P({X > a, Y ≥ b}C) = 1 - P({X > a}C {Y > b}C) (since (AB)C = AC BC) = 1 - P({X ≤ a} {Y ≤ b}) = 1 - [P(X ≤ a) + P(Y ≤ b) - P(X ≤ a, Y ≤ b)] (since P(AB) = P(A) + P(B) - P(AB)) = 1 - FX(a) - Fy(b) + F(a,b) | | | | | | | | | |
| P(a1 ≤ X ≤ a2, b1 ≤ Y ≤ b2) =  F(a2, b2) + F(a1, b1) - F(a1, b2) - F(a2, b1) | | | Proof. | | | | | | | | | |
| Suppose 2 balls are chosen w/o replacement from urn consisting of 4W, 6B balls. Let Xi = 1 if ith ball selected is white and 0 otherwise. Find joint pmf of (X1, X2) | | | |  |  |  | | --- | --- | --- | | X1 \ X2 | 0 | 1 | | 0 | 1/3 | 4/15 | | 1 | 4/15 | 2/15 |   P(X1 = 0, X2 = 0) = (6/10)(5/9) = 1/3...  pmf of X1. P(X1 = 0) = 1/3 + 4/15 = 3/5  P(X1 = 1) = 4/15 + 2/15 = 2/5  Can find pmf of X2 as well | | | | | | | | | |
| f(x, y) =  Find probs.  Find pdf of X and pdf of Y | | | P(X > 1, Y < 2) = = = = = -e-1 = -e-1[e-2y] = -e-1(e-4 - 1) = e-1(1-e-4)  P(X < Y) = = ... = 1/3. P(Y > 2) = = ... = e-4 | | | | | | | | | |
| fX(x) = = ... = . fY(y) = = ... = | | | | | | | | | |
| Joint pdf of X and Y, f(x,y) =  Find pdf of X/Y. Let W = X/y | | | | | | | | | | | FW(w) = P(W ≤ w) = P(X/Y ≤ w) = = = ... = 1 - , w > 0 (since x > 0, y > 0, x/y > 0). fW(w) = FW(w) = | |
| Indep r.v. | | | | | | | | | | | | |
| Suppose that n+m indep trials, with P(success) = p. Let X = num of successes in 1st n trials, Y = num of success in last m trials, Z = total success in n+m trials | | | | P(X = x, Y = y) = px(1-p)n-xpy(1-p)m-y = P(X = x)P(Y = y), 0 ≤ x ≤ n, 0 ≤ y ≤ m. So X and Y are indep  Z = X + Y. P(X = x, Z = z) = P(X = x, X + Y = z) = P(X = x, Y = z - x) = px(1-p)n-xpz-x(1-p)m-z+x. So P(X = x, Z = z) ≠ P(X = x)P(Z = z) = px(1-p)n-xpz(1-p)n+m-z. So X and Z are dependent | | | | | | | | |
| Num of ppl entering post office in a day is Poisson r.v. w paremeter . If P(male) = p, P(female) = 1-p. Show num of males and females entering office are indep Poisson r.v. w respective parameters p and (1-p) | | | | Let X = num of males entering, Y = num of females. Given X + Y ~ Poisson().  P(X + Y = i+j) = . P(X = i, Y = j|X + Y = i+j) = pi(1-p)j  P(X = i, Y = j) = P(X = i, Y = j|X + Y = i+j)P(X + Y = i+j) + P(X = i, Y = j|X + Y ≠ i+j)P(X + Y ≠ i+j) =  pi(1-p)j + 0 = = = P(X=i)P(Y=j)  So X and Y indep. P(X=i) = , i = 0,1,2,.... X ~ Poisson(p). Y ~ Poisson((1-p)) | | | | | | | | |
| Diagram  Description automatically generatedMan and woman meet. If each person independently arrives at a time uniformly distributed btw 12 and 1pm, find prob 1st to arrive has to wait longer than 10 mins. | | | | | | | | | | Let X = time past 12pm man arrives, Y = time past 12pm woman arrives. X ~ Uniform(0,60), Y ~ Uniform(0,60). f(x,y) = fx(x)fy(y) = (1/60)(1/60) = 1/602, 0 < x < 60, 0 < y < 60  P(Y > X + 10 or X > Y + 10) = P(Y > X + 10) + P(X > Y + 10) = 2P(Y > X + 10) (by symmetry) = 2 = 25/36 OR 2 | | |
| Buffon's needle problem. Table has equidistant parallel lines at distance D apart. Needle of length L, where L ≤ D, is randomly thrown on table. Prob that needle will intersect one of the lines? | | | | Needle will intersect if cos > X, where X = dist from middle of needle to nearest // line. Note X ~ Uniform(0, D/2), ~ Uniform(0, π/2) and X and are indep.  f(x,y) = fx(x)(y) = 1/(D/2) \* 1/(π/2) = 4/(Dπ), 0 ≤ x ≤ D/2, 0 ≤ y ≤ π/2  P(X < cos ) = = . Then π = =  By throwing needle N times, find num of times needle intersect line, then π = | | | | | | | | |
| X and Y indep iff their joint pdf/pmf can be expressed as f(x, y) = g(x)h(y), -∞ < x < ∞, -∞ < y < ∞ | | | | Proof. : X and Y indep f(x,y) = fx(x)fy(y). : f(x,y) = h(x)g(y). 1 = = = c1c2. fx(x) = = h(x)= c2h(x). Similarly, fy(y) = c1g(y). So, f(x,y) = h(x)g(y) = = fx(x)fy(y) (since c1c2 = 1). So X and Y indep. | | | | | | | | |
| f(x, y) = . X and Y indep? | | | | | | | f(x,y) = 6e-2xe-3yIx(x)Iy(y) = 6e-2xI(x) \* e-3yI(y) = g(x)h(y) for , where Ix(x) = , same for Iy(y). So X and Y indep since f(x, y) = g(x)h(y), \*\*-∞ < x < ∞, -∞ < y < ∞\*\* | | | | | |
| f(x, y) = . Are X and Y indep? | | | | f(x,y) ≠ h(x)g(y) for all x,y -∞ < x < ∞, -∞ < y < ∞. Define I(x,y) = =  f(x,y) = 24xyI(x,y) -∞ < x < ∞, -∞ < y < ∞ which cannot be factored as h(x)g(y), so not indep. | | | | | | | | |
| Let X1, X2, ... be seq of indep and identically distributed cts r.v. and suppose we observe these r.v. in seq. If Xn > Xi for each i = 1,2,...,n-1, then we say Xn is a record value. Let An = {Xn be record value}. Is An+1 indep of An? | | | | | | | | | E.g. P(A6) = any one of the 6 x's can be the largest, so P(A6) = 1/6  P(An+1|An) = P(An+1)? If it is, then indep. But this is diff to calculate  Consider P(An|An+1) = P(An) (since An+1 is a future event) = 1/n. So An is indep of An+1  By symmetric relation of independence, An+1 also indep of An | | | |
| Sum of indep r.v. | | | | | | | | | | | | |
| Diagram, schematic  Description automatically generatedDiagram  Description automatically generatedX ~ Uniform(0,1), Y ~ Uniform(0,1), X + Y ~ triangular | | | | | | | | fx(x) = , fy(y) = . f(x,y) = fx(x)fy(y) =  Let W = X+Y. Fw(w) = P(W ≤ w) = P(X + Y ≤ w)  For 0 < w < 1, P(X+Y ≤ w) = P(X ≤ w - y) = = = w2/2  Chart, line chart  Description automatically generatedFor 1 < w < 2, P(X + Y ≤ w) = 1 - P(X + Y > w) = = 2w - w2/2 - 1  FX+Y(w) = , fX+Y(w) = | | | | |
| 3. Zi ~ N(0, 1), i = 1,...,n ~ (chi-square w n deg of freedom)  Note pdf of Gamma(, ) =  = = | | | | | | | | Proof. Z ~ N(0,1). fZ(z) = , -∞ < z < ∞. Let Y = Z2.  FY(y) = P(Y ≤ y) = P(Z2 ≤ Y) = P(- ≤ z ≤ ) = P(Z ≤ ) - P(Z ≤ -) = FZ() - FZ(-).  fY(y) = FY(y) = fZ() - FZ(-) *­* = + = , y > 0. So Z2 ~ Gamma(, ). Using result 1, + + ... + ~ Gamma(, ) | | | | |
| 5. X ~ Poisson(), Y ~ Poisson() X + Y ~ Poisson( + ) | Proof. P(X + Y = n) = P(X = 0, Y = n) + P(X = 1, Y = n-1) + ... + P(X = n, Y = 0) = = (since indep) = = = = (binomial expansion) | | | | | | | | | | | |
| Conditional dist for discrete case | | | | | | | | | | | | |
| If X and Y are indep Poisson r.v. w parameters , . Conditional dist of X given X + Y = n? | | | | X ~ Poisson(), Y ~ Poisson(), X + Y ~ Poisson( + ).  P(X = k|X + Y = n) = = = (since indep) = = = = . X|X+Y=n ~ Binomial(n, ) | | | | | | | | |
| Conditional dist for cts case | | | | | | | | | | | | |
| Joint pdf of X and Y is f(x,y) = | | | 1 = ... c =  marginal pdf of y, fY(y) = = (y3 + y - y2/2), 0 < y < 1  conditional pdf of X given Y = y, fX|Y(x|y) = = = , 1-y < x < 1  fX|Y(x|) = = (x+), < x < 1. P(X > |Y = ) = = = 14/19  X and Y indep? f(x,y) = I(x,y), -∞ < x < ∞, -∞ < y < ∞, where I(x,y) =  Note f(x,y) ≠ h(x)g(y), so X and Y not indep | | | | | | | | | |
| Joint prob dist of Fn of r.v. | | | | | | | | | | | | |
| Let X1 and X2 be indep r.v., both uniformly dist on (0,1), i.e. = 1 on (0,1)  Find joint pdf of Y1 = X1 + X2 and Y2 = X1 - X2  Find pdf of Y1  Diagram  Description automatically generated | | | | y1 = x1 + x2, y2 = x1 - x2. x1 = (y1 + y2)/2, x2 = (y1 - y2)/2  J(x1, x2) = = = -2. (y1, y2) = (x1, x2) = 1\*  (y1, y2) =  For 0 < y1 < 1, = = y1. For 1 < y1 < 2, = = 2 - y1  = | | | | | | | | |
| =  Find joint pdf of Y1 = X1 + X2 and Y2 =  Find marginal pdf of Y2 | | | | y1 = x1 + x2 and y2 = . x1 = y1y2, x2 = y1(1-y2). J(x1, x2) = = =  (y1, y2) = (x1, x2) = = y1, y1 > 0, 0 < y2 < 1  (y2) = = = 1, 0 < y2 < 1 (by parts) | | | | | | | | |
| Extra | | | | | | | | | | | | |
| Joint pdf of X and Y is f(x,y) = c for 0 < x < 1 and 0 < y < 2 where c is a constant and zero otherwise. | | | | | | = 1. c = 1/2. P(X > 1/2) = = 3/4. P(X > Y) = = 1/4 | | | | | | |
| Suppose 3 balls are chosen w/o replacement from an urn consisting of 5W and 8R balls. Let Xi = 1 if ith ball chosen is W and 0 otherwise. Give joint pmf of X1, X2 and X1, X2, X3  Suppose now W balls are numbered, let Yi = 1 if the ith W ball is chosen and 0 otherwise. Find joint pmf of Y1, Y2 and Y1, Y2, Y3. | | | | | X: p(0,0) = = . p(0, 1) = p(1,0) = = . p(1,1) =  X: p(0,0,0) = . p(0,0,1) = p(0,1,0) = p(1,0,0) = . p(0,1,1) = p(1,0,1) = p(1,1,0) = . p(1,1,1) =  Y: p(0,0) = = . p(1,1) = = . p(0,1) = p(1,0) = = . Y: p(0,0,0) = = . p(0,0,1) = p(0,1,0) = p(1,0,0) = = . p(i,j,k) = = . p(1,1,1) = | | | | | | | |
| Joint pdf of X and Y is f(x,y) = (x2 + ), 0 < x < 1, 0 < y < 2. Verify is a joint density fn.  Compute density fn of X  P(X>Y). P(Y > 1/2 | X < 1/2). E(X). | | | | = 1. fX(x) = = , 0 < x < 1  P(X > Y) = = . P(Y > |X < ) = = =  E(X) = = = . | | | | | | | | |
| Suppose n points are indepently chosen at random on perimeter of circle, and we want the prob that they all lie in some semicircle.  Let P1,...,Pn denote the n points. Let A = {all points are contained in the same semicircle}. Ai = {all points lie in semicircle beginning at point Pi and gg clockwise for 180}, i = 1,...,n | | | | | | | | | | | | Express A in terms of Ai. A = (A is true as long as one of the Ai is true)  Are Ai mutually exclusive (ME)?. Yes  P(A) = P() = = (ME) = n |
| 3 pts, X1, X2, X3 are selected at random on line L. | | | | | | P(X2 lies btw X1 and X3) = 1/3 (by symmetry). Any of the 3 pts equally likely to be middle one | | | | | | |
| 2 pts are selected randomly on line of length L so as to be on opp sides of the midpoint of line. i.e. X and Y are indep r.v. and X ~ U(0,L/2), Y ~ U(L/2, L). | | | | Diagram  Description automatically generatedf(x) = 2/L, 0 < x < L/2. f(y) = 2/L, L/2 < y < L. f(x,y) = (2/L)(2/L) = 4/L2, 0 < x < L/2, L/2 < y < L  P(Y - X > L/3) = + = 7/9 | | | | | | | | |
| Show f(x,y) = 1/x, 0 < y < x < 1 is a joint density fn. Assume f is joint density fn of X,Y | | | | Diagram, schematic  Description automatically generated = 1. fY(y) = = -ln(y), 0 < y < 1. fX(x) = = 1, 0 < x < 1.  E(X) = = 1/2. E(Y) = = 1/2 (by parts) | | | | | | | | |
| Let f(x,y) = 24xy, 0 ≤ x ≤ 1, 0 ≤ y ≤ 1, 0 ≤ x + y ≤ 1 | | | | Show f(x,y) is a joint pdf. f(x,y) ≥ 0. AND = 1  fX(x) = = 12x(1-x)2, 0 < x < 1. E(X) = = 2/5  By symmetry, E(Y) = E(X) = 2/5 | | | | | | | | |
| Number of ppl that enter a store in a given hour is a Poisson r.v. w = 10. Compute conditional prob that at most 3 men enter store, given that 10 women entered in the hour. Assumptions? | | | | Let X = num of men who entered, Y = num of women who entered. X + Y ~ Poisson(10)  If we assume prob of men entering p = 1/2, women entering is 1-p, and X and Y are indep, then  X ~ Poisson(1x0\*1/2 = 5). Y ~ Poisson(5).  P(X ≤ 3|Y = 10) = P(X ≤ 3) (indep) = e-5 + + + | | | | | | | | |
| Joint density of X and Y, f(x,y) =  Are X and Y indep? What if f(x,y) = | | | | | | | | | | fX(x) = = xe-x = xe-x, x > 0  fY(y) = = e-y = e-y, y > 0. f(x,y) = xe-(x+y) = fX(x)fY(y). X and Y indep  fX(x) = = 2(1-x), 0 < x < 1. fY(y) = = 2y, 0 < y < 1  fX(x)fY(y) = 4y(1-x) ≠ 2 = f(x,y). X and Y not indep | | |
| Joint density fn of X and Y is f(x,y) = | | | | Are X and Y indep? No, cause f(x,y) cannot be written in the form f(x,y) = h(x)g(y)  Find density fn of X. fX(x) = = x + 1/2, 0 < x < 1  P(X+Y < 1) = = 1/3 | | | | | | | | |
| Consider indep trials each of which result in outcome i, i = 0,1,...,k w prob pi = = 1.  Let N denote num of trials needed to obtain outcome that is not equal to 0, and let X be that outcome. | | | | | | | | | P(N = n), n ≥ 1 = . P(X = j), j = 1,...,k =  Show P(N=n, X=j) = (outcome 0)n-1\*(outcome j) = = = P(N=n)P(X=j) | | | |
| Weekly sales at a restaurant is a normal r.v. w mean $2200 and s.d. $230.  Let W = X1 + X2, where X1 ~ N(2200, 2302), X2 ~ N(2200, 2302) | | P(2 weeks sales > 5000)? W ~ N(4400, 2(230)2). P(W > 5000) = P( > ) = P(Z > 1.8446) = 0.0326  P(weekly sales > 2000 in at least 2 of the next 3 weeks)? p = P(X > 2000) = P( > ) = P(Z > -0.87) = 0.8078. 3 weeks + 2weeks = p3 + 3p2(1-p) | | | | | | | | | | |
| 2 dice are rolled. Let X = largest value and Y = smallest value. Compute conditional mass fn of Y given X = i, for i = 1,2,...,6. Are X and Y indep?  X and Y are not indep. In particular, Y ≤ X | | | | For j = i, P(Y = i|X = i) = = . For j < i, P(Y = j|X = i) = =  For a fixed i, 1 = + P(Y = i|X = i) = + . P(X = i) = + 1/36. (Multiply P(X = i) on both sides). P(X = i) = (i-1)(2/36) + 1/36 = (2i-1)/36  P(Y = j|X = i) = | | | | | | | | |
| Joint density fn of X and Y, f(x,y) = xe-x(y+1), x > 0, y > 0  Find conditional density of X, given Y = y and that of Y, given X = x  Find density fn of Z = XY | | | | fY(y) = = , y > 0. f­X|Y(x|y) = = = (y+1)2xe-x(y+1), x > 0  fX(x) = = e-x, x > 0. f­Y|X(y|x) = = = xe-xy, y > 0  FZ(z) = P(Z ≤ z) = P(XY ≤ z) = = 1 - e-z, z > 0. fZ(z) = FZ(z) = e-z, z > 0 | | | | | | | | |
| Diagram  Description automatically generatedJoint density fn of X and Y, f(x,y) = c(x2- y2)e-x, 0 ≤ x < ∞, -x ≤ y ≤ x  Find conditional dist of Y given X = x | | | | fX(x) = = ce-xx3(). f­Y|X(y|x) = = = , -x ≤ y ≤ x  FY|X(a|x) = = = (x2a - + )  FY|X(y|x) = (x2y - + ), -x < y < x | | | | | | | | |
| If X and Y have joint density fn f(x,y) = , x ≥ 1, y ≥ 1  Compute joint density fn of U = XY, V = X/Y  What are the marginal densities?Diagram  Description automatically generated | | | | u = xy, v = x/y. y = . x = . J(x, y) = = = y() – x() = . |J(x,y)| = = 2v  (u, v) = (x, y) = = = , u ≥ v, uv ≥ 1 (since ≥ 1 and ≥ 1)  fU(u) = = ln u, u ≥ 1  For v > 1, fV(v) = = , v > 1  For v < 1, fV(v) = = , 0 < v < 1 | | | | | | | | |
| If X1 and X2 are indep exponential r.v. each w parameter , find the joint density fn of Y1 = X1 + X2 and Y2 = | | | | y1 = x1 + x2. y2 = . So x1 = ln y2. x2 = y1 - ln y2. J(x1, x2) = = = - = -y2  f(x1, x2) = (since indep), x1 > 0, x2 > 0  (y1, y2) = (x1, x2) = , y2 > 1, y1 > ln y2 | | | | | | | | |
| Suppose X and Y are indep geometric r.v. w param p. Without any computations, what do you think is value of P(X = i|X + Y = n)? | | | | Given 2nd success occur at nth trial, 1st success can occur at any of 1st n-1 trials w prob 1/(n-1)  P(X = i|X + Y = n) = = = = = | | | | | | | | |
| If X is exponential w rate , find P{[X] = n, X - [X] ≤ x}, where [x] is defined as largest integer ≤ x. Can you conclude that [X] and X are indep? | | | | | P{[X] = n, X - [X] ≤ x} = P(n < X < n+x) = = (1 – )  So indep | | | | | | | |

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| Expectation of Sums of r.v. | | | | | | | | | | | | | | | | | | |
| P(a ≤ X ≤ b) a ≤ E(X) ≤ b | | | | | | | | | | Proof (cts case). ≤ ≤ .  a ≤ E(X) ≤ b. a ≤ E(X) ≤ b | | | | | | | | |
| Diagram  Description automatically generated | | | | | | | | | | f(x,y) = , find E(XY)  E(XY) = = = = ... = 1/12 | | | | | | | | |
| If E(X) & E(Y) are finite, E(X + Y) = E(X) + E(Y) | Proof (cts case). E(X+Y) = = + = + = + = E(X) + E(Y) | | | | | | | | | | | | | | | | | |
| Let X1,..., Xn be indep and identically distributed r.v. having dist F(x) and E(Xi) =  If (sample mean) = E() = | | | | | | | | | | | | | | | | | E() = E() = E() = = = (n) = | |
| If X~Binomial(n,p), then E(X) = np | | | | | | | | | | X = X1 + X2 + ... + Xn where Xi = . E(Xi) = 1P(Xi = 1) + 0P(Xi = 0) = 1p = p  E(X) = E(X1) + ... + E(Xn) = np | | | | | | | | |
| Negative Binomial: If X is num of trials until total r successes obtained, E(X) = r/p | | | | | | | | | | X = X1 + X2 + ... + Xr, Xi = num of trials until next success ~ Geometric(p). E(Xi) = 1/p  E(X) = E(X1 ) + E(X2) + ... + E(Xr) = r/p | | | | | | | | |
| Hypergeometric: Take n balls from urn containing m W and N-m B balls, E(Num of white balls selected) = nm/N | | | | | | | | | | Let X = num of W balls selected, Yi = . X = Y1 + Y2 + ... + Yn. E(Yi) = 1P(Yi = 1) = m/N. E(X) = E(Y1) + ... + E(Yn) = nm/N | | | | | | | | |
| Hat throwing: E(num of ppl that select their own hat) = 1 | | | | | | | | | | Let X = num of matches, Xi = . X = X1 + ... + XN. P(Xi = 1) = 1/N  E(Xi) = 1P(Xi = 1) = 1/N. E(X) = N(1/N) = 1 | | | | | | | | |
| Coupon problem: E(num of coupons to be collected to obtain complete set) = 1 + + + ... + | | | | | | | Let X = num of coupons for complete set, Xi = num of additional coupons after i distinct types collected in order to obtain another distinct type, i = 0,1,2,...,N-1. X = X0 + X1 + ... + XN-1  X­0 = 1. X1 ~ Geometric(), X2 ~ Geometric(),... , XN-1 ~ Geometric(). E(X) = E(X0) + ... + E(XN-1) = 1 + + + ... + = 1 + + + ... + = N( + + + ... + 1) | | | | | | | | | | | |
| Covariance, Variance of Sums, Correlations | | | | | | | | | | | | | | | | | | |
| X and Y indep E[g(X)h(Y)] = E[g(X)]E[h(Y)] | | | | | | | | | | Proof (cts case). E[g(X)h(Y)] = = (since indep) = = E[g(X)]E[h(Y)] | | | | | | | | |
| Cov(X, Y) = E(XY) - E(X)E(Y) | | | | | | | | | | Proof. Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY - XE(Y) - YE(X) + E(X)E(Y)] = E(XY) - E[XE(Y)] - E[YE(X)] + E[E(X)E(Y)] = E(XY) - E(Y)E(X) - E(X)E(Y) + E(X)E(Y) = E(XY) - E(X)E(Y) | | | | | | | | |
| Cov() = | | | Proof. Let E(Xi) = , i = 1,...,n and E(Yj) = , j = 1,...,m. E[] = = . E[] = = . Cov() = E = E = E = = | | | | | | | | | | | | | | | |
| Var() = + 2 | | | | | | | | | | | | Proof. Var() = Cov(, ) (ii) = (iv) = + = + 2 (ii + i) | | | | | | |
| If X~Binomial(n,p) then Var(X) = np(1-p) | | | | | | | | | | Let X = num of success in n indep trials. Xi = . X = X1 + ... + Xn and Xi's are indep  E(Xi) = 1P(Xi = 1) = p. E(Xi2) = 12P(Xi = 1) = p. Var(Xi) = E(Xi2) - [E(X)]2 = p - p2 = p(1-p)  Var(X) = Var(X1 + ... + Xn) = Var(X1) + ... + Var(Xn) (indep) = np(1-p) | | | | | | | | |
| Let X1,...,Xn be indep and identically distributed r.v. each having expected value and var . Let = and s2 =  s2 is an unbiased estimator of | | | | | | Var() = Var() = Var() = (since Xi indep) = (n) =  (n-1)s2 = = = = + - = + n - = + n - = + n - = – n  E[(n-1)s2] = E[ – n]. (n-1)E(s2) = E[] – E[n] = – nE[] = – nVar() = n – n() = (n-1). So E(s2) = . | | | | | | | | | | | | |
| Diagram  Description automatically generated | | f(x,y) = , find Cov(X,Y)  E(X) = = 1/3. E(Y) = = 1/3. E(XY) = = 1/12  So Cov(X, Y) = E(XY) - E(X)E(Y) = 1/12 - (1/3)(1/3) = -1/36 < 0. As x incr, y decr | | | | | | | | | | | | | | | | |
| -1 ≤ (X, Y) ≤ 1  Note (X, Y) = = | | | | | | Proof. Suppose Var(X) = , Var(Y) = . 0 ≤ Var( + ) = Var() + Var() + 2Cov(, ) = Var(X) + Var(Y) + Cov(X, Y) = 1 + 1 + 2(X, Y) = 2[1+(X, Y)]. (X, Y) ≥ -1. Similarly, starting from 0 ≤ Var( – ), (X, Y) ≤ 1 | | | | | | | | | | | | |
| Let IA and Ib be indicator variables, IA = , IB = | | | | | | | E(IA) = 1P(A) + 0P(AC) = P(A). E(IB) = P(B). IAIB = . E(IAIB) = P(AB)  Cov(IA, IB) = E(IAIB) - E(IA)E(IB) = P(AB) - P(A)P(B) = P(B)[ - P(A)] = P(B)[P(A|B) - P(B)], i.e.cov = 0 if A indep of B | | | | | | | | | | | |
| Let X1 ,..., Xn be indep and identically distributed r.v. w variance . Show Cov(Xi - ,) = 0 | | | | | | | | | | Cov(Xi + (-),) = Cov(Xi, ) + Cov(-,) (iv) = Cov(Xi, ) - Cov(,) = Cov(Xi, ) - Var() = - Var() (iv) = Cov(Xi, Xi) - (since Xi, Xj indep, cov = 0) = Var(Xi) - = - = 0 | | | | | | | | |
| Conditional Expectation | | | | | | | | | | | | | | | | | | |
| X,Y ~ Geometric(p) and indep, find E(X|X+Y = n) | | | | | | | | | P(X = i|X+Y = n) = , i = 1,2,...,n-1. E(X|X+Y = n) = = = = | | | | | | | | | |
| f(x,y) = , find E(X|Y = y) | | | | | fY(y) = = = 2(1-y), 0 <y< 1. f­X|Y(x|y) = = = , 0 < x < 1-y. So X|Y = y ~ U(0, 1-y)  E(X|Y = y) = = , 0 < y < 1 | | | | | | | | | | | | | |
| Toss 2 dice. X, Y = largest, smallest value. find E(Y|X = 4) | | | | | | | | | | P(Y = 1, X = 4) = 2/7. P(Y = 2|X=4) = 2/7 = P(Y=3|X=4). P(Y=4|X=4) = 1/7  E(Y|X=4) = 1(2/7) + 2(2/7) + 3(2/7) + 4(1/7) = 16/7 | | | | | | | | |
| E(X) = | | | | | | | | Proof (discrete case). = = = = = = E(X) | | | | | | | | | | |
| Miner and 3 doors. 1st door lead to exit after 3 hrs. 2nd door lead to starting place after 5 hrs. 3rd door lead to starting place after 7 hours. Assume miner at all times equally likely to choose any door, expected time until exit? | | | | | | | | | | | | | | | Let X = amt of time until exit, Y = door chosen  E(X) = = E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3) = 3(1/3) + [5+E(X)](1/3) + [7+E(X)](1/3). So E(X) = 15 | | | |
| f(x,y) = ,  3 diff ways of finding E(X) | | | | | 1. E(X) = = 1/3. {E[g(X, Y)] = }  2. fX(x) = = 2(1-x), 0 < x < 1. E(X) = = 1/3. {E(X) = }  3. fY(y) = 2(1-y), 0 < y < 1. E(X|Y = y) = . E(X) = = 1/3. {E(X) = } | | | | | | | | | | | | | |
| Suppose num of ppl entering store on a given day is r.v. w mean 50. Suppose amt of money spent by customers is are indep r.v. w mean $8. Assume amt of money spent by customer is indep of total num of customers in store.  Find expected amt of money spent in store on a given day. | | | | | | | | | | | | | | Let N = num of customers that enter store, Xi = amt spend by ith customer  Total amt of money spent = which is a r.v.  E() = E(E()) = (cond expectation over all n) = (change of var) = (N and Xi are indep) = = (E(Xi) all same) = E(X1) = E(X1)E(N) = 8 \* 50 = 400 | | | | |
| P(E) =  P(E) = | | | | | | | | | | Proof. Let E denote any arbitrary event and X = . E(X) = 1P(E) + 0P(EC) = P(E)  E(X|Y=y) = 1P(X=1|Y=y) + 0P(X=0|Y=y) = P(X=1|Y=y) = P(E|Y=y)  E(X) = . P(E) = E(X) = | | | | | | | | |
| Diagram  Description automatically generatedf(x,y) = ,  find P(X < Y)  OR = 1/2 | | | | | | | | | | P(X < Y) = , where fY(y) = 2(1-y), 0 < y < 1. X|Y = y ~ U(0, 1-y) (pg 5), fX|Y(x|y) = , 0 < x < 1-y  For y < 1/2: P(X < y|Y = y) = = . For y > 1/2. P(X < y|Y = y) = 1 (since x + y < 1)  P(X < Y) = + = 1/2 | | | | | | | | |
| Suppose by any time t, num of ppl that have arrived at a train station is a Poisson r.v. w mean t. If train arrives at station at time (indep of ppl arrival) uniformly dist over (0,T). What is mean and var of num of ppl entering train? | | | | | | | | | | Let N(t) = num of arrivals by time t, Y = time train arrives  E(N(Y)|Y=t) = E(N(t)|Y=t) = E(N(t)) (since N(t) and Y are indep) = t.  E(N(Y)|Y) = Y, a r.v.. E(E(N(Y)|Y)) = E(Y) = E(Y) = (T/2)  Var(N(Y)) = E[Var(N(Y)|Y)] + Var[E(N(Y)|Y)]  Var(N(Y)|Y=t) = Var(N(t)|Y=t) = Var(N(t)) (N(t) and Y are indep) = t  Var(N(Y)|Y) = Y, a r.v.. E[Var(N(Y)|Y)] = E[Y] = E(Y) = (T/2)  Var[E(N(Y)|Y)] = Var(Y) = Var(Y) = (T2/12). Thus, Var(N(Y)) = (T/2) + (T2/12) | | | | | | | | |
| Moment Generating Functions | | | | | | | | | | | | | | | | | | |
| Mn(t) = E(XnetX), n ≥ 1 | | | | | | | | | | Proof. M(t) = E(etX). M'(t) = E(etX) = E(etX) = E(XetX). M'(0) = E(X)  M''(t) = E(XetX) = E(XetX) = E(X2etX). M''(0) = E(X2) | | | | | | | | |
| X~Binomial(n,p), M(t) = (pet + (1-p))n | | | | | | | M(t) = E(etX) = = = = (pet + (1-p))n (binomial expansion). M'(t) = n(pet + 1-p)n-1. M'(0) = E(X) = np. M''(t) = n(n-1)(pet + 1-p)n-2(pet)2 + n(pet + 1-p)n-1pet. M''(0) = E(X2) = n(n-1)p2 + np. Var(X) = E(X2) - [E(X)]2 = n(n-1)p2 + np - [np]2 = np(1-p) | | | | | | | | | | | |
| X~Poisson(), M(t) = exp[(et - 1)] | | | | | | | M(t) = E(etX) = = = = = (expansion of ex) = = exp[]. M'(0) = E(X) = . M''(0) = E(X2) = + . Var(X) = | | | | | | | | | | | |
| X~Exp(), M(t) = /( - t) | | | | | | | M(t) = E(etX) = = = = for t < =  M'(0) = E(X) = . M''(0) = E(X2) = . Var(X) = | | | | | | | | | | | |
| X~Normal(0,1), M(t) = | | | | M(t) = E(etX) = = = = (complete the sq) = = = (1) (since = pdf of N(t,1))  M'(0) = E(X) = 0. M''(0) = E(X2) = 1. Var(X) = 1 | | | | | | | | | | | | | | |
| X and Y indep MX+Y(t) = Mx(t)MY(t) | | | | | | | | | | Proof. MX+Y(t) = E[et(X+Y)] = E[etXetY] = E(etX)E(etY) (since X and Y indep) = Mx(t)MY(t) | | | | | | | | |
| If X and Y are indep r.v., X~Binomial(n,p), Y~Binomial(m,p), what is distribution of X + Y? | | | | | | | | | | MX(t) = (pet + (1-p))n. MY(t) = (pet + (1-p))m  Since X and Y indep, MX+Y(t) = Mx(t)MY(t) = (pet + (1-p))n(pet + (1-p))m = (pet + (1-p))n+m  Looking at the mgf, X + Y have dist Binomial(n+m, p) | | | | | | | | |
| If X = (, ) find mgf of X | | | | | | | | | | Z = ~ N(0,1). MZ(t) = = E(etZ) = E() = E() = E() = (let s = ) = E().  E() = . E() = . MX(t) = | | | | | | | | |
| Extra | | | | | | | | | | | | | | | | | | |
| Joint pdf of X and Y, f(x,y) = 2/3 for 0 < x < 1, 0 < y < 2, x < y and 0 otherwise. | | | | | | | | | | E(X) = = 4/9. [E(g(x,y) = ]  E(XY) = = 7/12  E(Y) = = 11/9. Cov(X, Y) = E(XY) - E(X)E(Y) = 13/324  fX(x) = = (4-2x)/3. fY|X(y|x) = f(x,y)/fX(x) = (2/3)/[(4-2x)/3] = 1/(2-x), x < y < 2  Then Y|X = x ~ U(x, 2). E(Y|X = x) = (x+2)/2. OR E(Y|X = x) = = (x+2)/2  Y|X = x ~ U(x, 2). Var(Y|X = x) = (2-x)2/12 | | | | | | | | |
| Z ~ N(0,1). MZ(t) = . Find dist of -Z | | | | | | | | | | M-Z(t) = E(et(-Z)) = E(e(-t)Z) = MZ(-t) = = . Thus, -Z ~ N(0,1) | | | | | | | | |
| Discrete r.v. X has pmf P(X = -1) = 1/4, P(X = 0) = 1/2, P(X = 1) = 1/4. Fing mgf of X. | | | | | | | | | | M(t) = = e-tP(X = -1) + e0tP(X = 0) = etP(X = 1) = (e2t + 2et + 1)/(4et) | | | | | | | | |
| Hospital is located at center of a square w length 3 miles. If accident occur within square, then hospital sends out an ambulance. The road network is rectangular, so the travel dist from hospital, whose coordinates are (0,0) to the point (x,y) is |x| + |y|. If an accident occurs at a pt that is uniformly distributed in the sq, find the expected travel dist of the ambulance. | | | | | | | | | | | | | | | | joint density (X, Y) at which accident occurs is f(x,y) = 1/9, -3/2 < x, y < 3/2 = f(x)f(y) where f(a) = 1/3, -3/2 < a < 3/2. Hence X and Y are indep and uniformly distributed on (-3/2, 3/2).  E(|X| + |Y|) = 2 = 3/2 | | |
| Suppose A and B each randomly and independently choose 3 out of 10 objects. Find the expected num of objects  a) chosen by both A and B  b) not chosen by either A or B  c) chosen by exactly one of A and B | | | | | | | | | | Let Xi = 1 if both choose item i, and 0 otherwise. Let Yi = 1 if neither A nor B choose item i and 0 otherwise. Let Wi = 1 if exactly one of A and B choose item i and 0 otherwise.  Let X = , Y = , W =  a) E(X) = = 10(3/10)2 = .9  b) E(Y) = = 10(7/10)2 = 4.9  c) Since X + Y + W = 10. E(W) = 10 - .9 - 4.9 = 4.2 OR = 10(2)(3/10)(7/10) | | | | | | | | |
| Cards are turned face up 1 at a time. If 1st card is ace, or 2nd a deuce, or 3rd a 3, or ... or 13th a King, or 14th an ace, and so on, we say that a match occurs. | | | | | | | | | | | | | Compute expected num of matches that occur  E(number of matches) = E, Ii = = 52E(Ii) = 52(1/13) = 4 | | | | | |
| Let X be a r.v. having finite expectation and variance , and let g be a twice differentiable fn. Shoe E(g(X)) ≈ g() + | | | | | | | | | | g(X) = g() + g'()(X - ) + g''() + ... ≈ g() + g'()(X - ) + g''()  E(g(X)) = E{g() + g'()(X - ) + g''() } = g() + g'()E(X - ) + g''() E =  g() + g'(){E(X) - } + = g() + | | | | | | | | |
| Let X­1, X2, ..., Xn be indep and identically distributed +ve r.v.. Find for k ≤ n, E | | | | | | | | | | 1 = E = E = = n. So = , then E = | | | | | | | | |
| If E(X) = 1 and Var(X) = 5, find E[(2+X)2] and Var(4 + 3X) | | | | | | | | | | E[(2+X)2] = E[X2 + 4X + 4] = E[X2] + 4E(X) + 4 = {Var(X) + [E(X)]2} + 4E(X) + 4 = 14  Var(4 + 3X) = 9E(X) = 45 | | | | | | | | |
| If 10 married couples are randomly seated at a round table, compute expected num and var of num of wives who are seated next to their husbands | | | | | | | | | | Let Xj = . E() = = 10[1\*P(Xj = 1)] = 10(2/19) = 20/19. (Since there are 2 ppl seated next to wife j, prob 1 of them is her husband is 2/19)  Var() = + 2Cov(Xi, Xj) = 10(2/19)(17/19) (var of Bernoulli) + 90[E(XiXj) - E(Xi)E(Xj)] = 340/361 + 90[P(Xi = 1, Xj = 1) - E(Xj)2] = 340/361 + 90[P(Xi = 1)P(Xj = 1|Xi = 1) - (2/19)2] = 340/361 + 90[(2/19)(2/18) - 4/361] (couple 1 next to e.o., couple 2 need to be tgt and only have 18 seats left to choose from) = 360/361 | | | | | | | | |
| Let X be num of 1's and Y be num of 2's that occur in n rolls of a fair die. Cov(X, Y)? | | | | | | | | | | Let Xi = , Yi = . Cov(Xi, Yj) = E[XiYj] - E(Xi)E(Yj).  If i = j: XiYj = XiYi = 0, since roll i is either 1, 2, or others. Cov(Xi, Yj) = -E(Xi)E(Yj) = -P(Xi = 1)P(Yj = 1) = -1/36  If i ≠ j, Cov(Xi, Yj) = 0. (indep since 2 diff rolls)  Cov(X, Y) = Cov = = = -n/36 | | | | | | | | |
| Joint density fn of X and Y is f(x, y) = , x > 0, y > 0  Find E(X), E(Y) and show Cov(X, Y) = 1 | | | | | | | | | | fY(y) = e-y = e-y, y > 0. Y ~ Exp(1). E(Y) = 1, Var(Y) = 1.  fX|Y(x|y) = f(x,y)/fY(y) = . Then X|Y=y ~ Exp(1/y). E(X|Y=y) = y. E(X) = E(E(X|Y)) = E(Y) = 1  Cov(X, Y) = E(XY) - E(X)E(Y) = E[E(XY|Y)] - 1 = E[YE(X|Y)] (Since Y is a "constant" inside) - 1 = E(Y2) = 2 - 1 = 1 | | | | | | | | |
| Pond contains 100 fish, of which 30 are carps. If 20 fish are caught, what are the mean and var of num of carp among these 20. | | | | | | | | | | | Let X be num of carps caught. X ~ HGeo(20, 100, 30). E(X) = 20\*30/100 = 6. Var(X) = (20\*30)(100-30)(100-20)/(1002\*(100-1)) = 112/33 | | | | | | | |
| If X and Y are identically distributed, not nexessarily indep, show Cov(X + Y, X - Y) = 0 | | | | | | | | | | | Cov(X+Y, X-Y) = Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y) = Var(X) - Var(Y) = 0 | | | | | | | |
| Joint density of X and Y is given by f(x, y) = , 0 < x < ∞, 0 < y < ∞  Compute E[X2|Y = y] | | | | | | | | | | fX|Y(x|y) = = = , 0 < x < ∞. X|Y = y ~ Exp(1/y). E(X|Y = y) = y.  E[X2|Y = y] = Var(X|Y = y) + [E(X|Y = y)2] = y2 + y2 = 2y2 | | | | | | | | |
| Joint density of X and Y is f(x, y) = , 0 < x < y, 0 < y < ∞. Compute E[X3|Y = y] | | | | | | | | | | fX|Y(x|y) = = 1/y, 0 < x < y. E[X3|Y = y] = = y3/4 | | | | | | | | |
| Expected num of accidents per week at an industrial plant is 5. Suppose num of workers injured in each accident are indep r.v. w common mean of 2.5. If num of workers injured in each accident is indep of num of accident occuring, compute expected num of workers injured in a week. | | | | | | | | | | | | | | | | | | Let N be num of accidents, Xj be num of workers in accident j  E(X1 + X2 + ... + XN) = E[E(X1 + ... + XN|N)] (since N is a r.v. as well) = E(2.5\*N) = 2.5E(N) = 12.5 |
| Type *i* light bulbs fn for a random amt of time w mean and sd , i = 1,2. A light bulb randomly chosen from a bin of bulbs is a type 1 bulb w prob p, type 2 bulb w prob 1-p. Let X denote lifetime of bulb. Find E(X), Var(X) | | | | | | | | | | | | | | E(X) = E(X|type 1)p + E(X|type 2)(1-p) = p + (1-p). Let I be r.v. denoting type of light bulb  Var(X) = E[Var(X|I)] + Var[E(X|I)] = E[] + Var() = p + (1-p) + {E() - [E()]2} = p + (1-p) + {p + (1-p) - [p + (1-p)]2} | | | | |
| Num of accidents a person has in a given year is a Poisson r.v. w mean . Suppose value of is 2 for 60% of pop and 3 for other 40%. If person is chosen at random, prob that he will have 0 accidents, exactly 3 accidents in a year. What is the conditional prob that he will have 3 accidents in a given year, given he has no accidents in the previous year? | | | | | | | | | | | | | | | | P(0 accidents) = .6e-2 + .4e-3  P(3 accidents) = .6e-2(23/3!) + .4e-3(33/3!)  P(3 accidents|0) = = (since accidents in previous year don't affect curr year | | |
| Mgf of X is MX(t) = exp(2et - 2) and Y is MY(t) = . If X and Y are indep, what are P(X + Y = 2), P(XY = 0), E(XY) | | | | | | | | | | X is poisson w = 2. Y is Binomial w param (10, 3/4)  P(X + Y = 2) = P(X=0)P(Y=2) + P(X = 1)P(Y=1) + P(X=2)P(Y=0)  P(XY=0) = P(X=0) + P(Y=0) - P(X=0, Y=0)  E(XY) = E(X)E(Y) = 2\*10\*3/4 = 15 | | | | | | | | |
| Joint density of X and Y is f(x,y) = , 0 < y < ∞, -∞ < x < ∞. Compute joint mgf of X and Y. Compute individual mgf | | | | | | | | | | Note that Y is exp w rate 1, and given Y, X is normal w var 1  E[etX+sY] = E[etX+sY|Y] = esYE[etX|Y] = esY.  E[etX+sY] = E{E[etX+sY|Y]} = E{esY} = E[e(s+t)Y] =, s + t < 1  E(etX) = , t < 1 (let s = 0). E(esY) = , s < 1 | | | | | | | | |

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| Markov's Inequality. If X is a r.v. that takes only nonnegative values, then for any a > 0, P(X ≥ a) ≤ | | | | Proof. For. a > 0, let I = . Note I ≤ X/a. E(I) ≤ E(X)/a  E(I) = 1P(X ≥ a) + 0P(X < a) = P(X ≥ a). So P(X ≥ a) ≤ E(X)/a | | | | | |
| Chebyshev's Inequality. If X is a r.v. w finite mean and var , then for any value of k > 0, P(|X -| ≥ k) ≤ | | | | | (X - )2 ≥ 0. P[(X - )2 ≥ k2] ≤ (from Markov's Inequality)  P(|X-| ≥ k) ≤ (since E = var(X) = ) | | | | |
| Let X have pdf f(x) =  Find P(|X| ≥ 3/2) exactly and approximately using Chebyshev's inequality | | E(X) = 0 (since symmetric). Var(X) = = = 1  P(|X| > 3/2) = 1 - = 0.134  P(|X| > 3/2) = P(|X-0| > 3/2) ≤ = 0.444 (Chebyshev's) | | | | | | | |
| If Var(X) = 0, then P(X = E[X]) = 1 | | Proof. Let = E(X). Using Chebyshev's inequality, for any n ≥ 1, P(|X-| ≥ ) ≤ = 0  So P(|X-| ≥ ) = 0. 0 = = P() = P(X ≠ ). So P(X = ) = 1 | | | | | | | |
| Weak law of large nums. Let X1, X2,... be a seq of indep and identically distributed r.v. each having finite mean E[Xi] = . Then for any > 0, P 0 as n ∞ | | | | | | Proof. = . Var() = .  Using Chebyshev's inequality, P ≤ 0 as n ∞ | | | |
| Let X1, X2, ... be a seq of r.v. s.t. Xn ~ N(). If Xn X, what is the dist of X?  Note X~N(, ). M(t) = | | | | | | | | | Xn ~ N(). (t) = = mgf of N(, ). So X ~ N(, ). |
| CLT. Let X1, X2,... be a seq of indep and identically distributed r.v. each having mean and var . Then tends to standard normal as n ∞.  Note (Var(X) = E(X2) - [E(X)]2. 1 = E(X2) - 0.) | | Proof. Assume = 0, = 1. = = + ... +  M = = ... (since Xi are indep)  = E = E = = . So M =  Let L(t) = log[MX(t)] where MX(t) = E[etX]. Then L(0) = log[MX(0)] = log 1 = 0  L'(t) = . L'(0) = = = 0. L''(t) = . L''(0) = = E(X2) = 1.  So L() = log. = . M = = =  Need to show = = . Same as showing = t2/2  = = (L'Hopital's rule) = = = = t2/2 | | | | | | | |
| Suppose a seq of indep trials is performed. Let E be a fixed event and prob occur is P(E).  Let Xi = . Find | | | | | | | Using strong law of large nums, E(X1) with prob 1  E(X1) = 1P(E) + 0P(EC) = P(E)  In other words, P(E) with prob 1 | | |
| Chernoff bounds. P(X ≥ a) ≤ e-taM(t) for all t > 0. | | | Proof. P(X ≥ a) = P(tX ≥ ta) for t > 0 = P(etX ≥ eta) ≤ E(etX)/eta (Markov's inequality) = e-taM(t) for all t > 0. | | | | | | |
| Z ~ N(0,1), find Chernoff bound for Z | P(Z ≥ a) ≤ e-taMZ(t) for t > 0 = e-ta = for t > 0  h(t) = t2/2 - ta. h'(t) = t-a. h'(t) = 0, then t = a. h''(t) = 1 > 0 (min value). So P(Z ≥ a) ≤ (tightest bound) | | | | | | | | |
| Jensen's inequality  If f(x) is a convex fn, then E[f(X)] ≥ f(E(X)), if E(X) exists and is finite | | Proof. Let = E(X). Taylor's series expansion: f(x) = f() + f'()(x-) + where (x, )  ≥ 0 since convex fn. So f(x) ≥ f() + f'()(x-). f(X) ≥ f() + f'()(X-). E(f(X)) ≥ E{f() + f'()(X-)} = f() + f'()(E(X)-) = f(E(X)) + f'()(E(X)-E(X)) = f(E(X)) | | | | | | | |
| Let X be a positive r.v.. Show E(1/X) ≥ 1/E(X) | | f(x) = 1/x. f'(x) = -x-1. f''(x) = x-2 > 0 for all x > 0. f is convex fn.  By Jensen's inequality, E(1/X) ≥ 1/E(X) | | | | | | | |
| Let X be a r.v. w P(X ≤ 0) = 0. i.e. X is +ve.  Show P(X ≥ 2) ≤ 1/2 where = E(X) | | = ≥ ≥ (since smallest value x can take is ) = P(X ≥ )  P(X ≥ 2) ≤ 1/2 | | | | | | | |
| Extra | | | | | | | | | |
| Let X1, ..., X20 be indep Poisson r.v. w mean 1. Use Markov inequality to obtain a bound on P(). Use clt to approximate P(). | | | | | | | | P() ≤ E()/15 = 20/15  Sum ~ N(20, 20). P() = P() = .8428 | |
| A die is continually rolled until total sum of all rolls exceeds 300. Prob that at least 80 rolls are necessary? | | | | | If Xi is outcome on ith roll, then E(Xi) = 7/2, Var(Xi) = 35/12.  P(at least 80 rolls are needed) = P(sum ≤ 300) = P() = 0.9430 (normal approx) | | | | |
| If X is a gamma r.v. w param (n, 1), how large does n need to be s.t. P() < .01 | | Gamma(n,1) = sum of n indep exponential variables w rate 1, thus X has mean n, var n.  P() = P() = 2P(Z > .01)  2P(Z > .01) < .01. P(Z > .01) < .005. Using normal table, .01 = 2.58. n = 2582 | | | | | | | |